An Intrinsic Approach to Lichnerowicz Conjecture

Akhil Ranjan

February 7, 2008

Abstract

In this paper we give a proof of Lichnerowicz Conjecture for compact simply connected manifolds which is intrinsic in the sense that it avoids the *Nice Embeddings* into eigen spaces of the Laplacian. Even if one wants to use these embeddings this paper gives a more streamlined proof.

Keywords: Harmonic manifolds, Blaschke manifold, mean curvature, Jacobi differential equation, Ricci curvature, compact rank one symmetric spaces, nice embeddings.

1   Introduction

The object of this paper is to present an intrinsic proof of the Lichnerowicz's conjecture for the compact simply connected harmonic manifolds. For the definition of harmonic manifolds see [1]. One of characterisations is that the geodesic spheres around any point have constant mean curvature depending only on the radius of the sphere. It suffices to consider small values of the radii. Lichnerowicz showed that for dimension less than or equal to 4, such a manifold must be either flat or a locally symmetric space of rank one (see [4], [1]). He quite naturally asked whether the same was true in higher dimensions. Great progress was made in the case of compact simply connected harmonic manifolds,

---

a detailed account of which is given in Besse's book [1]. It is shown that these are all Blaschke manifolds and their Ricci tensor is proportional to the metric tensor or in other words they are Einstien manifolds. From topological point of view each such manifold has its (integral) cohomology ring isomorphic to precisely one of the compact rank one symmetric space to be referred to as its *model CROSS* henceforth. This result is due to Allemigeon. A particularly striking discovery about compact harmonic spaces is a family of isometric minimal immersions into the round spheres in eigen-spaces of the Laplacian acting on the space of square-integrable functions. Moreover, any two geodesics were shown to be congruent to each other under some Euclidean isometry. These are now known as *Besse's Nice Embeddings.* In 1990 Szabo [9] successfully used them along with other known facts about harmonic manifolds to answer Lichnerowicz's query affirmatively for compact simply connected harmonic manifolds. In contrast to compact case, Damek and Ricci in 1992 [3] produced a family of examples of homogeneous harmonic manifolds which are not locally symmetric. In Szabo's paper the key point was to show that the volume function of a compact simply connected harmonic manifold when expressed in terms of normal coordinates coincided with that of its *model CROSS*. To this end he goes through the following steps:

1. He establishes what he calls *basic commutativity in harmonic spaces.* This implies in conjunction with Allamigeon's theorem that for any point p on the manifold and any eigen value λ of the Laplacian there exist eigen-functions which depend only on radial distance from p. Moreover, starting with any eigen function and averageing over geodesic spheres around p we get such a radial eigen function.

2. By moving the point p along a geodesic and averageing in the said manner we get a parallelly displaced family of functions in the eigen-space which is finite dimensional. This along with the fact that each geodesic is periodic of period *assumed* to be 2π enables one to conclude that the *radial* eigen functions alluded to above are polynomials in cosine of the radial distance.

3. At this stage the *nice embeddings* are used to pin down the volume function in geodesic normal coordinates.

4. Finally it follows that the first radial eigen function is linear of the form $A\cos r + B$ and studying the nice embedding in the first eigen-space one shows symmetry easily.

In this paper we show that the *nice embeddings* can be avoided in the step (3) above. Steps (1) and (2) do not require them anyway. Our proof of step (3) can be regarded as a streamlined version of that given by Szabo. As for the last step one can either use *nice embeddings* or the partial solution to a problem of Antonio Ros about the first eigen value of P-manifolds given in [6]. In the latter case one has worked wholly within the manifold thereby proving Szabo's theorem intrinsically.

2   Laplacian on radial functions

Let $\Theta(r)$ denote the volume function on a simply connected compact harmonic manifold M in terms of normal coordinates and $\sigma(r)$ the mean curvature of any geodesic sphere of radius r. It is easily shown that

$$\frac{\Theta'(r)}{\Theta(r)} \;=\; \sigma(r)$$

Further for a point p and an eigen-value λ of the Laplacian Δ, let u be an eigen function which depends only on radial distance r from p. As shown by Szabo ([9],p.8,eq.(2.1)) u satifies the following ODE

$$u'' + \sigma(r)u' + \lambda u = 0 \tag{2.1}$$

Here $'$ means derivative with respect to r. We would like to study how closely Θ and σ agree with their anologues in its *model CROSS*. Let us first define the volume function on all of *real* line as follows. Consider a geodesic γ through a point p. Let $J_2, ..., J_d$ be the Jacobi fields along γ which vanish at $\gamma(0)$ and whose derivatives at $\gamma(0)$ form an orthonormal basis along with $\gamma'(0)$. Let $E_1, ..., E_d$ be parallel translation of the above orthonormal basis along γ, E_1 being $\gamma'(r)$. Now set

$$\Theta(r) = < J_2 \wedge ... \wedge J_d \, , \; E_2 \wedge ... \wedge E_d > (r)$$

. By virtue of it being a Blaschke manifold, Θ when considered as a function on whole of the real line has the following properties :

1. It is periodic of period 2π.

2. It has zeroes of order $k - 1$ at $r \;=\; n\pi$ for n any odd integer and zeroes of order $d - 1$ at $r = n\pi$ for n any even integer. Here d is the dimension of M and k is the degree of the generator of the cohomology ring of M.

3

3. $\Theta(r) = (-1)^{d-1}\Theta(-r)$

This clearly allows us to write

$$\Theta(r) = e^{\alpha(\cos r)}\sin^{d-1}(r/2)\cos^{k-1}(r/2)$$

or $\Theta(r) = e^{\alpha(\cos r)}\Theta_0(r)$ where $\Theta_0(r)$ is the volume function of the *model CROSS* and α is a smooth (actually analytic) function on [-1,1] with $\alpha(1) = 0$. Hence $\sigma(r) = \sigma_0(r) - \sin(r)\alpha'(\cos r)$ since $\sigma = \frac{\Theta'}{\Theta}$.

Caution: The conventional volume function is the absolute value of the one we have defined. They both agree within the *injectivity radius* i.e. for $0 \le r \le \pi$. For $0 < r < \pi$, an easy calculation gives that

$$\sigma_0(r) = \frac{1}{2\sin r}[(d-1)(1+\cos r) - (k-1)(1-\cos r)]$$

By Lemma 4.2 of [9], u in eq.(2.1) is of the form $u = f(\cos r)$ for some *polynomial f*. Inserting all this data in 2.1 and setting $\cos r = x$ we see that f satisfies the following

$$(1-x^2)f'' - [\frac{d}{2}(1+x) - \frac{k}{2}(1-x) + (1-x^2)\alpha'(x)]f' + \lambda f = 0, -1 \le x \le 1. \quad (2.2)$$

In the above equation $'$ denotes derivative w.r.t. x.

3 Jacobi Differential Equation

The differential equation

$$(1-x^2)u'' - [(1+b)(1+x) - (1+a)(1-x)]u' + \lambda u = 0 \quad (3.3)$$

has been studied classically as a (singular) Sturm-Liouville equation on [-1,1] and it is known that for a and b in $(-1,\infty)$ and under natural boundary conditions (u bounded as $|x| \to 1$) solutions exist for $\lambda = n(n+a+b+1), n \in I\!\!N$ and for each such value of λ, u is a polynomial of degree n. Moreover, u is unique upto a scalar multiple. In fact these polynomials form a complete orthogonal system in $L^2([-1,1], \rho dx)$ where $\rho(x) = (1+x)^a(1-x)^b$ is the weight function. These are known as Jacobi polynomials. (See [2] p. 289). This differential equation is known as Jacobi differential equation with parameters a and b. We assume that $a, b > -1$.

In this section we consider a *perturbed* Jacobi equation where we have an extra term of the form $(1-x^2)\delta(x)$ as a coefficient of u', δ being a continuous function on [-1,1].

Comparing with the corresponding equation satisfied by the polynomial f in the previous section we easily see that $1 + b = \frac{d}{2}$, $1 + a = \frac{k}{2}$ and $\delta = \alpha'$ We also know that k cannot exceed d and can only take values in $2, 4, 8, d$, hence $a, b > -1$ is clearly true. Now we are ready to state our main theorem.

Theorem 3.1 *If the perturbed Jacobi differential equation*

$$(1 - x^2)u'' - [(1 + b)(1 + x) - (1 + a)(1 - x) + (1 - x^2)\delta(x)]u' + \lambda u \ = \ 0 \qquad (3.4)$$

admits a nonconstant polynomial as a solution for some value of λ, then the perturbation term δ must vanish identically.

Corollary 3.1 *The perturbation term α' in 2.2 vainshes. Consequently α is identically zero and hence $\sigma = \sigma_0$ as well as $\Theta = \Theta_0$.*

The proof of the above will be broken into two lemmas. Let P be a polynomial which we assume to be nonconstant and monic which satisfies 3.4 for a suitable λ.

Lemma 3.1 *δ must be a rational function with the degree of the numerator being strictly less than that of the denominator.*

Proof:

$$\delta = \frac{LP + \lambda P}{(1 - x^2)P'}$$

where

$$L \ = \ (1 - x^2)\frac{d^2}{dx^2} - [(1 + b)(1 + x) - (1 + a)(1 + x)]\frac{d}{dx}$$

is the Jacobi differential operator (with parameters a and b). Clearly both numerator and denominator of δ are polynomials with denominator nonvanishing and of degree strictly more than that of the numerator. Hence the claim. ∎

Lemma 3.2 *Let $\delta = \frac{p}{q}$ as a quotient of relatively prime polynomials with q being monic. Then*

1. *All the roots of q are simple and in $\mathbb{C} \setminus [-1, 1]$.*

5

2. $q|P'$ and $q|P$.

3. Let $q = \prod(x - \beta_i)$ and m_i be the multiplicity of $\beta_i \in P$, then $m_i \geq 2$.

4. If we put $q_1 = \prod(x - \beta_i)^{m_i-1}$, then $\delta = \frac{q'_1}{q_1}$.

5. ± 1 cannot be roots of P.

6. Roots of P not common with those of q are all simple.

Proof: Let $P = \prod(x - \beta_i)^{m_i}$ where β_i are distinct complex numbers and m_i are natural numbers which are nonzero. Let

$$v = \frac{P'}{P} = \sum \frac{m_i}{x - \beta_i} \tag{3.5}$$

Then v satisfies the Riccati equation (see [2] p. 124)

$$v' + v^2 = [\frac{1+b}{1-x} - \frac{1+a}{1+x} + \delta(x)]v - \frac{\lambda}{1-x^2} \tag{3.6}$$

From 3.5 we get

$$v' + v^2 = \sum_i \frac{m_i^2 - m_i}{(x - \beta_i)^2} + \sum_{i \neq j} \frac{2m_i m_j}{(\beta_i - \beta_j)(x - \beta_i)} \tag{3.7}$$

In the above equation we have expanded the *cross terms* occuring in v^2 into partial fractions. Now let $q = \prod(x - \alpha_j)^{r_j}$ where α_j are distinct complex numbers and $r_j \geq 1$. Since p and q are relatively prime, if we expand $\delta = \frac{p}{q}$ in partial fractions, $\frac{1}{(x-\alpha_j)^{r_j}}$ will survive for each j. They will continue to survive after multiplication by $v = \sum \frac{m_i}{x-\beta_i}$ and further expansion into partial fractions. Now if we compare the rhs of 3.6 and 3.7 after expanding into partial fractions we find that we must have $\{\alpha_j\} \subset \{\beta_i\}$ and $r_j = 1$ for each j. This proves that the roots of q are simple. Since $\delta = \frac{p}{q}$ is continuous on [-1,1] roots of q must be away from [-1,1]. This poves the first assertion.

$q|P$ is clear from above. To show that $q|P'$, we note that $LP + \lambda P = \frac{p(1-x^2)P'}{q}$ is a polynomial. Hence $q|P'$ since it is relatively prime to p, $1 - x$, and $1 + x$. This gives the second claim.

Put $S = \{\beta_i\}, S' = \{\alpha_j\}$, then $S' \subset S$. Also put $S'' = S \setminus S'$. We can then write $q = \prod_{S'}(x - \beta_j)$ and hence $\delta = \sum_{S'} \frac{c_i}{x-\beta_i}$ where c_i are nonzero numbers. Coming back to the third statement, let us compare the coefficient of $\frac{1}{(x-\beta_i)^2}$ in the rhs of 3.6 and 3.7 (after partial fractions) for $\beta_i \neq \pm 1$. We see that

$$c_i \, m_i \;\; = \;\; m_i^2 \; - \; m_i \text{ for } i \text{ s.t } \beta_i \; \in \; S' \text{ and } m_i^2 - m_i = 0 \;\; \text{if} \;\; \beta_i \in S'' \setminus \{\pm 1\}$$

6

From this we can conclude that

$$c_i = m_i - 1 \text{ for } i \ \ s.t. \ \beta_i \in S' \text{ and}$$

$$m_i = 1 \text{ for } i \ s.t. \ \beta_i \in \ S'' \setminus \{\pm 1\} \text{ (since } m_i \neq 0 \text{ foreach } i).$$

$$\frac{p}{q} = \sum_{S'} \frac{m_i - 1}{x - \beta_i}$$

The first of these shows that $m_i \geq 2$ for $\beta_i \in S'$ because $c_i \neq 0$ and the second one can be rewritten as $\frac{p}{q} = \frac{q_1'}{q_1}$, where $q_1 = \prod_{S'}(x - \beta_i)^{m_i - 1}$. These are just the third and the fourth assertions.

For the fifth claim, write

$$P(x) = \prod_S (x - \beta_i)^{m_i} = (x-1)^A (x+1)^B \prod_{S'} (x - \beta_i)^{m_i} \prod_{S'' \setminus \{\pm 1\}} (x - \beta_i)$$

. Comparing coefficients of $(x-1)^{-2} and (x+1)^{-2}$ we see that

$$A^2 - A = -(1 + b)A \text{ and } B^2 - B = -(1 + a)B$$

Since a and b are more than -1 and A and B are natural numbers we get $A = B = 0$. Thus $S = S' \bigcup S''$; the set of roots of P, is disjoint from $\{\pm 1\}$

Finally, $S'' \setminus \{\pm 1\} = S''$ so that for $\beta_i \in \ S''$ we have $m_i = 1$ which is the sixth assertion. ∎

4 Proof of theorem 3.1

We have the following facts: $P(x) = \prod_{S'}(x - \beta_i)^{m_i} \prod_{S''}(x - \beta_i), m_i \geq 2. q_1(x) = \prod_{S'}(x - \beta_i)^{m_i - 1}$. Clearly, $q_1 | P'$. Let $\frac{P'}{q_1} = n \prod_j (x - \gamma_j)^{M_j} = R(x)$, say. (Here $n = degP$.) Then γ_j are those roots are P' which are not common with those of P. This is so because the roots of P in S'' are all simple. Hence

$$\{\gamma_j\} \bigcap S = \emptyset = S \bigcap \{\pm 1\}. \tag{4.8}$$

Substituting the expressions obtained for P, P' and δ in the equation 3.4, dividing by $(1 - x^2)P'$ and simplifying we get

$$\sum \frac{M_j}{x - \gamma_j} + \frac{a+1}{x+1} + \frac{b+1}{x+1} = -\frac{\lambda \prod_S (x - \beta_i)}{n \prod (x - \gamma_j)^{M_j}} \tag{4.9}$$

7

Putting $x = \beta \in S$ and using 4.8 we find that

$$\sum_j \frac{M_j}{\beta - \gamma_j} + \frac{a+1}{\beta+1} + \frac{b+1}{\beta-1} = 0, \quad \text{for each such} \quad \beta \tag{4.10}$$

This in turn implies that $\beta \in conv\{\gamma_j, \pm 1\}$ (where $conv$ denotes the *convex hull*). Or $S \subset conv\{\gamma_j, \pm 1\}$. On the other hand by Lucas' theorem $conv\{\gamma_j\} \subset convS$. Now the argument of Lemma 4.6 ([9], p 23) goes through and shows that $S \subset (-1, 1)$. Also as observed earlier S' is disjoint from [-1,1]. Therefore, S' must be empty so that q is constant 1 and hence $p = \delta$ vanishes identically as $deg(p) < deg(q)$. ∎

5 Proof of the conjecture

Let us first recall a theorem of Antonio Ros ([7],Theorem 4.2, p.402)

Theorem 5.2 *Let M be an n-dimensional $P_{2\pi}$-manifold and suppose that the Ricci tensor, S, and the metric, $\langle,\rangle$, on M verify the relation $S \geq k \langle,\rangle$, where k is a real constant. Let λ_1 be the first eigenvalue of the Laplacian of M. Then we have*

$$\lambda_1 \geq \frac{1}{3}(2k + n + 2)$$

.

He also remarked that for *CROSSES* the equality holds. He naturally asked as to what restrictions apply to M if equality held.

As a partial answer to the above question the following theorem has been proved ([6]) :

Theorem 5.3 *If equality holds in the Ros' estimate for λ_1 of a P-manifold and if M admits a corresponding eigen-function without saddle points, then M is a CROSS.*

Also see [5] for another related result.

Proof of the Conjecture: Now from the corollary 3.1 to our main theorem $\Theta = \Theta_0$ and hence $Ric_M = Ric_0$ and $\lambda_1(M) = \lambda_1(M_0)$ where M_0 denotes the *model CROSS*. Moreover, from any point on M the first radial eigen-function is of the form $\cos r + C$ and hence without saddle points. The proof of the Lichnerowicz conjecture is now complete.

References

[1] A. L. Besse, *Manifolds all of whose geodesics are closed*, Springer, Berlin, 1978.

[2] G. Birkhoff and G. C. Rota, *Ordinary differential equations*, John Wiley and Sons, 1989.

[3] E. Damek and F. Ricci, *A class of nonsymmetric harmonic Riemmanian spaces*, Bulletin AMS, **27** (1992), 139-142.

[4] A. Lichnerowicz, *Sur les espaces riemanniens complétement harmoniques*, Bull. Soc. Math. France, **72** (1944), 146-168.

[5] A. Ranjan and G. Santhanam, *A Generalisation of Obata's theorem*, Preprint, 1995.

[6] A. Ranjan and G. Santhanam, *The first eigen value of P-manifolds*, Preprint, 1995.

[7] A. Ros, *Eigenvalue Inequalities for Minimal Submanifolds and P-Manifolds*, Math. Zeit.,**187** (1984), 393-404.

[8] K. Sakamoto, *Helical immersions into a unit sphere*, Math. Ann., **261** (1982), 63-80.

[9] Z. I. Szabo, *The Lichnerowicz conjecture on Harmonic Manifolds*, J. Differential Geometry, **31** (1990), 1-28.

Department of Mathematics

Indian Institute of Technology

Mumbai 400076, INDIA

email: aranjan@ganit.math.iitb.ernet.in